PMM U.S.S.R.,Vol.52,No.4,pp.413-420,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ON THE SYMMETRY GROUPS OF DYNAMIC SYSTEMS*

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The existence of vector fields which commute with the vector field of the initial system and are defined in the entire phase space is discussed. The phase fluxes of these fields are well-known to be symmetry groups of a dynamic system, since they map the set of all its solutions into itself. Obstacles to the existence of non-trivial symmetry groups are the generation of a large number of non-degenerate periodic solutions, and the transversal intersection of asymptotic surfaces. The symmetry groups of systems of a "normal" type, which play an important part in perturbation theory, are examined in detail. The general results are applied, in particular, to Hamiltonian systems. It is shown that the equations of rotation of a heavy asymmetric rigid body with a fixed point do not have a non-trivial symmetry group if the centre of mass of the body is not the same as the point of suspension. In particular, there is no supplementary many-valued analytic integral which is independent of the classical energy and area integrals.

1. Symmetry groups. Consider the dynamic system given by the differential equation

$$dx/dt = v(x) \tag{1.1}$$

The vector field u that commutes with the field v (i.e. $[u, v] \equiv 0$, where $[,]$ is the vector field commutator) is called a symmetry field of system (1.1). The phase flux of the system

$$dx/d\tau = u(x) \tag{1.2}$$

which is a one-parameter group of mappings g_u^τ, maps a solution of system (1.1) into a solution of the system.

A dynamic system becomes easier to study if it has a symmetry group. By factorizing the group g_u with respect to the orbits, the order of system (1.1) can be reduced by one. This operation can be realized, at least locally, in any sufficiently small neighbourhood of a non-singular point of field u. Admittedly, reduction of the order rests constructively on finding the orbits (trajectories) of the system of differential Eqs.(1.2). Assume that there is a further symmetry field w, and that $[u, w] = \lambda w$. Then the order of system (1.1) can be reduced by two. Finally, if the system of n equations has a resolvent symmetry group of dimensionality $n-1$, then the system can be integrated in quadratures (Lie's theorem /1, 2/).

By the rectification theorem, in a small neighbourhood of a non-singular point of the vector field v, system (1.1) has an n-dimensional Abelian group of symmetries. Thus the existence of a smooth field of symmetries represents an interesting problem either in the neighbourhood of equilibrium, or in the entire phase space.

Let us quote two simple examples of dynamic systems which have non-trivial analytic symmetry fields, but do not have variable continuous integrals.

1) Consider conditionally periodic motion in the n-dimensional torus $\mathbf{T}^n = \{x_1, \ldots, x_n \bmod 2\pi\}$, specified by the system $x_i^\cdot = \omega_i$ with constant frequencies ω_i which are independent over the ring of integers. This system is ergodic in $\mathbf{T}^n$ and therefore has no variable continuous integrals. Yet any constant vector field in $\mathbf{T}^n$ is a symmetry field.

2) Let $v(x) = Ax$, where all the eigenvalues of the operator A lie in the left (or right) half-plane. since the equilibrium $x=0$ as $t \to +\infty$ (or as $t \to -\infty$) is asymptotically stable, the corresponding system (1.1) has no variable first continuous integrals. Yet $u \equiv x$ is a symmetry field; it generates the group of extensions $x \mapsto e^\tau x, \tau \in \mathbf{R}$.

In a Hamiltonian system the presence of a first integral F implies the presence of a symmetry group: the Hamiltonian vector field with Hamiltonian F is a symmetry field. This remark can be generalized. Let ω be a closed 1-form in the phase space of a system with Hamiltonian H. Locally $\omega = dF$, so that we can associate with the form ω a locally Hamiltonian vector field with Hamilton function F. If H and F are in involution, this field is a symmetry field of the initial Hamiltonian system. We can call the form ω (or the many-valued function F) the many-valued integral of the system with Hamiltonian H.

*Prikl.Matem.Mekhan.,52,4,531-541,1988

Let us give an example of a many-valued integral. For this, consider the motion of a charged particle over the plane torus $T^2 = \{x, y \bmod 2\pi\}$ in a constant magnetic field. The equations of motion

$$x^{\cdot\cdot} - \alpha y^{\cdot} = 0, \quad y^{\cdot\cdot} + \alpha x^{\cdot} = 0; \quad \alpha = \mathrm{const}$$

have two integrals, linear in the velocity: $x^{\cdot} - \alpha y$ and $y^{\cdot} + \alpha x$, which are many-valued functions in the phase space $T^2 \times R^2$.

We consider below the existence of symmetry fields which are defined in the entire phase space. Since λv, $\lambda = \mathrm{const}$, is a trivial symmetry field, we need to introduce the assumption that the fields u and v are linearly independent. Note that, if $u = \lambda(x) v$ and $[u, v] \equiv 0$, then λ is a first integral of system (1.1).

2. Non-degenerate closed trajectories.

Let v be an analytic vector field in the analytic manifold M^n. The periodic trajectory γ is called non-degenerate if $n-1$ of its multiplicators differ from unity. Let Γ denote the union of all non-degenerate closed trajectories of system (1.1). We call Γ a key set if any analytic function in M which vanishes in Γ is identically zero throughout M.

Theorem 1. If Γ is a key set, then any analytic symmetry field u of system (1.1) is linearly dependent on v at all points M. If, moreover, $v \neq 0$, then $u = \lambda v$, $\lambda = \mathrm{const}$.

Proof. Let γ be a non-degenerate closed trajectory. Then system (1.1) will have no other closed trajectories with a similar period in a small neighbourhood of γ If u is a symmetry field, then $g_u^\tau(\gamma)$ is a closed trajectory of (1.1), whose period differs little from the period of γ for small τ. Hence $g_u^\tau(\gamma) \equiv \gamma$ for all τ, so that the vectors u, v are linearly dependent at points of γ. This last property holds everywhere in Γ. Now let Ω be any analytic 2-form in M. Since $\Omega(u, v)$ is an analytic function in M, which vanishes at points of Γ, then $\Omega(u, v) \equiv 0$. We use the following fact: let Ω_0 be a given outer form at the point $x_0 \in M$; there is an analytic differential form Ω_x in M, which is identical with Ω_0 when $x = x_0$. Hence we obtain the linear dependence of fields u, v at all points of M. If $v \neq 0$, then $u = \lambda(x) v$, where λ is an analytic function in M, which is an integral of (1.1) (see Par.1). We know that $d\lambda = 0$ at points of the set Γ /3/ (Par.64). Since Γ is a key set, then $\lambda = \mathrm{const}$, which it was required to prove.

As an example, take a compact surface M, and assume that (1.1) is a U-system /4/. We know that all the periodic trajectories are hyperbolic (and hence non-degenerate), and that the set Γ is everywhere dense in M /4/. Hence the U-system does not have even non-trivial continuous symmetry fields. In particular, a geodesic flux on a compact manifold with negative curvature has no many-valued integrals.

A related example is the special case of the bounded three-body problem when two heavy bodies rotate their common centre of mass in an elliptic orbit with non-zero eccentricity, while the third body, of negligible mass, always moves in a straight line orthogonal to the plane of the heavy body orbits /5/. The extended phase space of this non-autonomous system is three-dimensional. From the results of /5/ we see that the set of hyperbolic periodic trajectories is a key set. The system therefore has no non-trivial symmetry group and in particular, has no many-valued analytic integral. It was noted in /5/ that no one-valued integral exists. We can similarly prove that there is no analytic symmetry group on energy surfaces with large negative energy in the case of the plane circular bounded three-body problem. The preparatory results required are proved in /6/ by methods of symbolic dynamics.

3. Splitting of asymptotic surfaces.

Assume that M^3 is a three-dimensional analytic manifold, and that the analytic vector field v has no equilibrium positions on it. Assume that there are two hyperbolic periodic trajectories γ_1 and γ_2. Denote by Λ_1^+ (Λ_2^-) the stable (unstable) asymptotic surface of trajectory γ_1 (γ_2). These surfaces are regular and analytic.

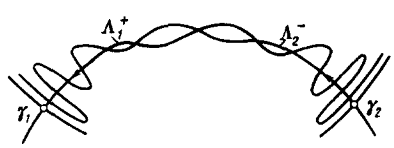

Fig.1

Theorem 2. Assume that Λ_1^+ and Λ_2^- intersect and are not identical with a set of points in M. Then, system (1.1) has only trivial analytic symmetry fields: $u = \lambda v$, $\lambda = \mathrm{const}$.

The scheme of the proof is as follows. The intersection $\Lambda_1^+ \cap \Lambda_2^-$ consists of the trajectories of system (1.1) which approach γ_1 (γ_2) without limit as $t \to +\infty$ $(t \to -\infty)$. Transformations of the group g_u map these doubly asymptotic trajectories into themselves. On the doubly asymptotic trajectories, the fields u and v are linearly dependent. Otherwise, Λ_1^+ and Λ_2^- would intersect along two-dimensional analytic areas, and hence would coincide, since they are regular and analytic. Since the asymptotic surfaces Λ_1^+ and Λ_2^- oscillate (Fig.1) and the fields u, v are analytic, the fields are linearly dependent at all points of M. It only remains to note that, under the assumptions of Theorem 2, system (1.1) has no variable analytic integrals in M /7/.

We apply Theorem 2 to the problem of the rotation of a heavy rigid body about a fixed point. We shall regard the problem as a perturbation of the integrable Euler-Poinsot problem. The small parameter ε is the product of the body weight and the distance from the centre of mass to the suspension point. By excluding the group of rotations about the vertical and fixing the value of the constant area, we reduce the problem to study of a Hamiltonian system with two degrees of freedom.

Theorem 3. If the body is dynamically asymmetric, the Hamiltonian system with two degrees of freedom has, for small non-zero values of ε, only a trivial analytic symmetry group.

Corollary. Under our assumptions, the equations have no many-valued analytic integral which is independent of the energy integral.

This assertion strengthens the well-known result of /8, 9/ on the non-existence of single-valued analytic first integrals.

To prove Theorem 3, we fix a positive value of the energy integral. With $\varepsilon = 0$, there are two periodic motions of the hyperbolic type (constant rotations of the body about the central axis of intertia) in the three-dimensional integral manifold. Their stable and unstable asymptotic surfaces are double. It was shown in /10, 9, 11/ that, for small $\varepsilon \neq 0$, the surfaces split up, and some of the disturbed asymptotic surfaces always intersect without coinciding. Hence Theorem 2 is applicable.

The asymptotic surfaces often intersect transversally in problems of Hamiltonian mechanics, such as the oscillations of a pendulum with a vibrating point of suspension, Kirchhoff's problem on the motion of a rigid body in a fluid, and the problem of four vortices, etc., see /7/. Theorem 2 holds in all these cases. It would be interesting to extend Theorem 2 to the multidimensional case. We need to speak here of the simultaneous existence of several symmetry fields, the number of which is the same as the number of degrees of freedom.

4. Perturbation theory. We consider the existence of a symmetry group for a system of differential equations of "normal" form which is often encountered in applications:

$$y_j^{\cdot} = \varepsilon F_j + \ldots, \quad x_k^{\cdot} = \omega_k + \varepsilon\Phi_k + \ldots; \quad 1 \leqslant j \leqslant m, \quad 1 \leqslant k \leqslant n \tag{4.1}$$

Here, the frequencies ω_k depend only on the slow variables y, while x are angular variables (the right-hand sides are 2π-periodic with respect to all x_k), and ε is a small parameter; the dots denote terms of order 2 or higher in ε.

We shall consider the symmetries of (4.1) generated by the system of equations

$$y_j' = Y_j^0 + \varepsilon Y_j^1 + \ldots, \quad x_k' = X_k^0 + \varepsilon X_k^1 + \ldots \tag{4.2}$$

The coefficients Y_j^λ and X_k^μ are assumed to be 2π-periodic in the coordinates $x_1, \ldots, x_k$. In other words, for the field v_ε we find the symmetries u_ε which are analytic in ε.

We shall confine ourselves to the "non-degenerate" case, when the following conditions hold:

1) $n \geqslant m$ and the rank of the matrix $\| \partial\omega_k/\partial y_j \|$ is almost everywhere equal to m;

2) if $\Sigma \omega_k(y) \alpha_k \equiv 0$ with certain integers α_k, then all the $\alpha_k = 0$.

For instance, with $m = 1$ these conditions certainly hold when the curve $y \mapsto \omega(y)$ is regular and intersects transversally the resonant surfaces $\Sigma\omega_k\alpha_k = 0$ $(\alpha \in \mathbf{Z}^n)$. If $m = n$, the coditions for non-degeneracy reduce to the single condition: almost everywhere $\det \| \partial\omega_k/\partial y_j \| \neq 0$.

We assume that all the functions encountered below are analytic.

First we put $\varepsilon = 0$ and find all the symmetry fields of the undisturbed integrable system. It can be shown that the commutation condition for the phase fluxes of the undisturbed systems (4.1) and (4.2) is equivalent to the series of equations

$$\sum \frac{\partial Y_j^\circ}{\partial x_l} \omega_l = 0, \quad 1 \leqslant j \leqslant m \tag{4.3}$$

$$\sum \frac{\partial \omega_k}{\partial y_j} Y_j^\circ = \sum \frac{\partial X_k^\circ}{\partial x_l} \omega_l, \quad 1 \leqslant k \leqslant n \tag{4.4}$$

Lemma 1. If the system is non-degenerate, then $Y_j^\circ \equiv 0$, and the X_k° are independent of $x_1, \ldots, x_n$.

Proof. We solve Eqs. (4.3), (4.4) by Fourier's method. We put

$$Y_j^\circ = \Sigma \zeta_\alpha(y) e^{i(\alpha, x)}, \quad (\alpha, x) = \Sigma \alpha_k x_k$$

We then find from (4.3) that $(\alpha, y)\zeta_\alpha \equiv 0$. Since the undisturbed system is non-degenerate by hypothesis and there are no zero divisors in the ring of analytic functions, then $\zeta_\alpha = 0$

for $\alpha \neq 0$. Thus the functions Y_j° depend only on y. On then averaging both sides of Eqs. (4.4) with respect to $x_1, \ldots, x_n$, we arrive at the relation

$$\Sigma \frac{\partial \omega_k}{\partial y_j} Y_j^\circ = 0$$

Since rank $\|\partial\omega_k/\partial y_j\| = m \leqslant n$, by hypothesis, then $Y_j^\circ \equiv 0$. We then find from (4.4) that the functions X_k° depend only on the slow variables.

We put

$$F_j = \Sigma f_\alpha^j (y) e^{i(\alpha, x)}, \quad Y_j^1 = \Sigma g_\alpha^j (y) e^{i(\alpha, x)}$$

From the commutation conditions for the phase fluxes of systems (4.1) and (4.2), we can obtain, to a first approximation in ε, the equation

$$(\alpha, \omega) g_\alpha = (\alpha, X^\circ) f_\alpha, \quad \alpha \in \mathbf{Z}^n \tag{4.5}$$

Here, $X^\circ, f_\alpha, g_\alpha$ are vectors with components $X_k^\circ, f_\alpha^j, g_\alpha^j$. Essential use is made of Lemma 1 when obtaining (4.5).

We introduce the resonant set $\mathbf{K}$, consisting of all the points $y \in \mathbf{R}^m$ for which there exist $n - 1$ linearly independent vectors $\alpha, \alpha', \ldots, \in \mathbf{Z}^n$ such that $(\alpha, \omega(y)) = (\alpha', \omega(y)) = \ldots = 0$ and $f_\alpha(y) \neq 0, f_{\alpha'}(y) \neq 0, \ldots$.

Lemma 2. Assume that, in a bounded open domain $D \subset \mathbf{R}^m = \{y\}$ the frequency vector $\omega \neq 0$ and $\mathbf{K} \cap D$ is a key set. Then, there is an analytic function ξ_0 such that $X^\circ = \xi_0 \omega$.

For, the vectors X° and ω are linearly dependent at points of the set $\mathbf{K}$. Note that, in the typical case, $\mathbf{K}$ is everywhere dense in $\mathbf{R}^m$.

We replace X° in (4.5) by the vector field $\xi_0\omega$ and use the inequality $(\alpha, \omega) \not\equiv 0$. Then $g_\alpha = \xi_0 f_\alpha$ for all $\alpha \neq 0$. We put

$$\Phi_k = \Sigma \varphi_\alpha^k (y) e^{i(\alpha, x)}, \quad X_k^1 = \Sigma \Psi_\alpha^k (y) e^{i(\alpha, x)}$$

Since systems (4.1) and (4.2) commute, we obtain to a first approximation in ε the chain of relations

$$i(\alpha, \omega) \Psi_\alpha^k = (\alpha, X^\circ) \varphi_\alpha^k + \sum \frac{\partial \omega_k}{\partial y_j} g_\alpha^j - \Sigma \frac{\partial X_k^\circ}{\partial y_j} f_\alpha^j, \qquad \alpha \in \mathbf{Z} \tag{4.6}$$

Assuming in (4.6) that $y \in \mathbf{K}$, $X_k^\circ = \xi_0 \omega_k$, and using the inequality $\omega \neq 0$, we arrive at the relation

$$\Sigma \frac{\partial \xi_0}{\partial y_j} f_\alpha^j = 0 \tag{4.7}$$

We introduce the distribution of $(m - 1)$-dimensional planes generated by the linearly independent vectors $f_\alpha, f_{\alpha'}, \ldots$ at the points $y \in \mathbf{K}$. Let η be the vectors at points $y \in \mathbf{K}$ which are orthogonal to these hyperplanes. The vector field η is defined in the "discontinuous" set $\mathbf{K}$. Let $\mathbf{K}'$ denote the set of points of $\mathbf{K}$ at which the vector function η is not continuous.

Lemma 3. Assume that $\mathbf{K}' \cap D$ is a key set. Then $\xi_0 = \text{const}$.

For, by (4.7), the vector $\partial\xi_0/\partial y$ is parallel to η at all points of $\mathbf{K}$. Since the field $\partial\xi_0/\partial y$ is continuous, it must vanish at points where η is discontinuous.

Lemma 3 is a crude sufficient condition for the function ξ_0 to be constant: if the field η is continued up to an analytic field throughout $\mathbf{R}^m$, the corresponding distribution of hyperplanes will in general not be integrable. The meaning of condition (4.7) becomes clear if we take the problem of whether system (3.1) has a single-valued analytic integral in the form of the series

$$H = H_0 + \varepsilon H_1 + \ldots \tag{4.8}$$

The functions H_0 and H_1 satisfy the equations

$$\Sigma \frac{\partial H_0}{\partial x_k} \omega_k = 0, \quad \Sigma \frac{\partial H_0}{\partial y_j} F_j + \Sigma \frac{\partial H_1}{\partial x_k} \omega_k = 0 \tag{4.9}$$

It follows from the first equation, using the non-degeneracy condition, that H_0 depends only on the variables y. Putting

$$H_1 = \Sigma h_\alpha (y) e^{i(\alpha, x)}$$

we obtain from the second equation of (4.9) the chain of relations

$$\Sigma \frac{\partial H_0}{\partial y_j} f_\alpha^j + i(\alpha, \omega) h_\alpha = 0, \quad \alpha \in \mathbf{Z}^n$$

At points of the set $\mathbf{K}$, the function H_0 satisfies (4.7). If the conditions of Lemma 3 hold, then $H_0 \equiv \text{const}$. It can be shown by induction in the same way that all the $H_s \equiv \text{const}$. Thus Lemma 3 gives a sufficient condition for system (4.1) to have no variable analytic integrals of the type (4.8).

Theorem 4. Let the conditions of Lemmas 1-3 hold. Then $u_\varepsilon = \xi v_\varepsilon$, where the function ξ depends only on ε.

For, by Lemmas 1-3, $u_0 = \xi_0 v_0$, $\xi_0 = \text{const}$. Hence the vector field $w_\varepsilon = (u_\varepsilon - \xi_0 v_\varepsilon)/\varepsilon$ is likewise an analytic symmetry field. Again from Lemmas 1-3, we have $w_0 = \xi_1 v_0$, $\xi_1 = \text{const}$, and so on. We arrive as a result at the equation $u_\varepsilon = \xi v_\varepsilon$, where $\xi = \xi_0 + \varepsilon\xi_1 + \ldots$.

5. Application of Hamilton's equations.

We consider the Hamiltonian system

$$x_k^{\cdot} = \partial H/\partial y_k, \quad y_k^{\cdot} = -\partial H/\partial x_k; \quad k = 1, \ldots, n \qquad (5.1)$$
$$H = H_0(y_1, \ldots, y_n) + \varepsilon H_1(y_1, \ldots, y_n, x_1, \ldots, x_n) + o(\varepsilon)$$

with a Hamiltonian which is analytic and 2π-periodic in x. In this case, non-degeneracy means that the Hessian of H_0 with respect to the variables y is non-zero, while the condition $\omega \neq 0$ is equilvalent to the absence of critical points of the function H_0. Comparing (4.1) and (5.1), we see that $F_j = -\partial H_1/\partial x_j$.

We Fourier-expand the disturbing function

$$H_1 = \Sigma h_\alpha(y)\, e^{i(\alpha, x)}$$

Then, $f_\alpha = -ih_\alpha\alpha$. In the present problem, $\mathbf{K}$ is the set of all points $y \in \mathbf{R}^n$ for which there exist $n-1$ linearly independent vectors $\alpha, \alpha', \ldots \in \mathbf{Z}^n$ such that $(\alpha, \omega(y)) = (\alpha', \omega(y)) = \ldots = 0$ and $h_\alpha(y) \neq 0$, $h_{\alpha'}(y) \neq 0, \ldots$. As distinct from the general case, system (4.7) here always has a non-trivial solution: it is satisfied by any analytic function of H_0. Hence Theorem 4 does not hold for Hamiltonian systems.

Let v_ε be a Hamiltonian vector field (5.1).

Theorem 5. Assume that y° is a non-critical point of the function H_0 and that $\det \| \partial^2 H_0/\partial y^2 \| \neq 0$ at this point. Assume also that, in any small neighbourhood U of the point y°, $\mathbf{K}$ is a key set. Then, in the domain $U \times \mathbf{T}^n \subset \mathbf{R}^n \times \mathbf{T}^n$, we have the equation $u_\varepsilon = \Phi(H, \varepsilon)\, v_\varepsilon$, where Φ is an analytic function.

Proof. By Lemmas 1 and 2, $u_0 = \xi_0 v_0$, where ξ_0 is an analytic integral of the undisturbed system, which depends only on y. We know /3, 7/ that the functions ξ_0 and H_0 are dependent. Since there are no critical points of H_0 in the small neighbourhood U, we have $\xi_0 = \Phi_0(H_0)$ in this domain, by the implicit function theorem, where Φ_0 is an analytic function, see /7/. Consequently, the vector field $w_\varepsilon = (u_\varepsilon - \Phi_0(H)\, v_\varepsilon)/\varepsilon$ is again an analytic symmetry field. Similarly, $w_0 = \Phi_1(H_0)\, v_0$ and so on. As a result we arrive at the equation $u^\cdot = \Phi(H, \varepsilon)\, v_\varepsilon$, where $\Phi = \Phi_0 + \varepsilon\Phi_1 + \ldots$

According to Poincaré /3, 7/, the conditions of Theorem 5 guarantee that there is no auxiliary analytic integral which is independent of the energy integral. Our theorem therefore strengthens Poincaré's result under the same assumptions. Our theorem is applicable to many problems of Hamiltonian mechanics, and in particular, to the plane circular bounded three-body problem (compare /3, 7/).

6. The case of a meagre resonant set.

Theorem 5 is not applicable in cases when the resonant set $\mathbf{K}$ consists of only a finite number of surfaces. Here again, it is sometimes possible to prove that there are no non-trivial symmetry groups.

Take the Hamiltonian system (5.1) with two degrees of freedom, in which $H = H_0 + \varepsilon H_1$; $H_0 = (\Sigma a_{ij} y_i y_j)/2$ is a positive definite quadratic form with constant coefficients, while the disturbing function

$$H_1 = \Sigma h_\alpha e^{i(\alpha, x)}, \quad h_\alpha = \text{const},$$

is a trigonometric polynomial. We introduce the finite set $\mathbf{M} = \{\alpha \in \mathbf{Z}^2 \colon h_\alpha \neq 0\}$, invariant under the involution $\alpha \mapsto -\alpha$. We also introduce the scalar product, given by $\langle \xi, \eta \rangle = \Sigma a_{ij} \xi_i \eta_j$.

Theorem 6. The Hamiltonian system with Hamiltonian $H_0 + \varepsilon H_1$ has a non-trivial symmetry group if and only if the points of $\mathbf{M}$ are located on $d \leqslant 2$ straight lines which intersect orthogonally (in the metric $\langle , \rangle$) at the origin.

The conditions of Theorem 6 are obviously sufficient: the Hamiltonian system has an auxiliary polynomial integral in the momenta of not higher than the second degree. The corresponding Hamiltonian field is the required symmetry field. The necessity is proved by means of the results of /12/, where a detailed analysis is given of the infinite number of steps of perturbation theory for system with Hamiltonian $H_0 + \varepsilon H_1$.

Instead of the laborious formal proof of Theorem 6, we shall consider here a special case which demonstrates the obstacles of a dynamic kind to the existence of a non-trivial symmetry group. Let $E(\mathbf{M})$ be the convex hull of the set $\mathbf{M}$, which is a convex polygon. Let α and α' be adjacent vertices of $E(\mathbf{M})$, where $\langle\alpha, \alpha'\rangle > 0$. Note that, since $\mathbf{M}$ is invariant under the involution $\alpha \mapsto -\alpha$, there will always be two vertices α and α' for which $\langle\alpha, \alpha'\rangle \geqslant 0$. Assume also that, for the infinite set of integers $m = 0, 1, 2, \ldots$, the components of the integer-valued vectors $m\alpha + \alpha'$ are relatively prime.

It was shown in /12/ that, under these assumptions, pairs of non-degenerate periodic solutions are generated on the two-dimensional resonant tori $y = y^\circ$, $x \bmod 2\pi$, $\langle m\alpha + \alpha', y^\circ\rangle = 0$, $y^\circ \neq 0$. In accordance with Para.1, the fields u_ε and v_ε are linearly dependent on the trajectories of these solutions. By the continuity, the fields u_0 and v_0 are linearly dependent on the "generating" periodic solutions which lie on the undisturbed resonant torus $y = y^\circ$. Since the undisturbed system is non-degenerate, the fields u_0 and v_0 are independent of the angular variables x. Consequently, they are linearly dependent at the point y°. There are such points on an infinity of different straight lines $\langle m\alpha + \alpha', y\rangle = 0$ which pass through the origin. The set of these lines forms a key set. Hence it follows that the vector fields u_0 and v_0 are linearly dependent at all points $y \in \mathbf{R}^2$. The proof can be completed by using the arguments of Para.5: if $y^\circ \neq 0$ is a point on the limit line $\langle\alpha, y\rangle = 0$ and U is a small neighbourhood of it, then we have $u_\varepsilon = \Phi(H, \varepsilon)\, v_\varepsilon$ in the domain $U \times T^2$, where Φ is an analytic function.

Theorem 6 also holds for Hamiltonian systems with $n > 2$ degrees of freedom, though we are speaking here of the existence of n symmetry fields $u_\varepsilon^1, \ldots, u_\varepsilon^n$, which are independent almost everywhere for $\varepsilon = 0$. Note also that the condition of Theorem 6 is the criterion for the existence of a supplementary single-valued integral, analytic in ε /12/.

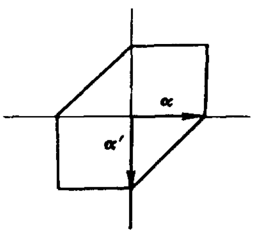

Fig.2

As an example, consider the motion of three particles of unit mass in a circle of unit radius, which are mutually elastically attracted or repelled. Let x_1, x_2, x_3 be the angular coordinates of the particles, and y_1, y_2, y_3 be their momenta. The Hamiltonian is

$$H = {}^1/_2\,(y_1^2 + y_2^2 + y_3^2) + \varepsilon\cos(x_1 - x_2) + \varepsilon\cos(x_2 - x_3) + \varepsilon\cos(x_3 - x_1)$$

Here, ε is a small coefficient of elastic interaction; it is negative (positive) in the case of attraction (repulsion). Apart from the energy integral, the equations of motion have the momentum integral $y_1 + y_2 + y_3$. We reduce the order of the system by means of the canonical transformation

$$q_1 = x_1 - x_2,\quad q_2 = x_2 - x_3,\quad q_3 = x_1 + x_2 + x_3$$
$$y_1 = p_1 + p_3,\quad y_2 = -p_1 + p_2 + p_3,\quad y_3 = -p_2 + p_3$$

In the new variables p, q, the integral p_3 is cyclical; we put $p_3 = 0$. We write the Hamiltonian of the reduced system:

$$H = p_1^2 - p_1p_2 + p_2^2 + \varepsilon\cos q_1 + \varepsilon\cos a_2 + \varepsilon\cos(q_1 + q_2) \tag{6.1}$$

The convex hull of the set $\mathbf{M}$ is shown in Fig.2. As the vertices of $E(\mathbf{M})$ we take the two vectors $\alpha = (1,0)$ and $\alpha' = (0, -1)$. Clearly, the components of the vector $m\alpha + \alpha'$ are prime and $\langle\alpha, \alpha'\rangle > 0$. Consequently, the system with Hamiltonian (6.1) has no non-trivial analytic symmetry fields. In particular, there are no many-valued integrals which are analytic in ε and independent of the energy integral. A difficulty is the fact that system (6.1) has an infinity of different families of non-degenerate long-periodic solutions.

7. Some generalizations. Assume that the fields u and v satisfy the relation

$$[u, v] = \mu v + \nu u \tag{7.1}$$

where μ and ν are constants. In the neighbourhood of a non-singular point of the field u we can use a local theorem on the rectification of trajectories and reduce Eqs.(1.2) to the form

$$dx_1/d\tau = \ldots = dx_{n-1}/d\tau = 0,\quad dx_n/d\tau = 1$$

If the general solution of system (1.2) is known, this reduction can be realized explicitly. In the variables $x_1, \ldots, x_n$ the commutation relation (7.1) is equivalent to the series of equations

$$\partial v_i/\partial x_n = \mu v_i,\quad i < n,\quad \partial v_n/\partial x_n = \mu v_n + \nu \tag{7.2}$$

where v_i are the components of the field v. From (7.2) we obtain

$$v_i = e^{\mu x_n} v_i^\circ,\quad i < n;\quad v_n = e^{\mu x_n} v_n^\circ - \nu/\mu \tag{7.3}$$

where the functions $v_i^\circ\ (1 \leqslant i \leqslant n)$ are independent of the coordinates x_n. We make the time replacement $dt = e^{\mu x_n} ds$ and write the first $n - 1$ equations of system (1.1), where the prime denotes differentiation with respect to s:

$$x_1' = v_1^\circ, \ldots, x_{n-1}' = v_{n-1}^\circ \tag{7.4}$$

This closed system of differential equations can be regarded as the result of reducing the order of the initial system (1.1).

Assertion 1. If the general solution of system (7.4) is known, Eq.(1.1) can be integrated in quadratures.

Since v_n° is independent of x_n, for the proof we only require to integrate the equation

$$x_n' = -\nu\mu^{-1}e^{-\mu x_n} + f \tag{7.5}$$

where f is a known function of s, see (7.3). By the replacement $z = e^{\mu x_n}$, we can reduce Eq.(7.5) to the form $z' = \mu fz - \nu$, which is easily integrated. The variables x_i are thus found explicitly as functions of s. In order to express the x_i in terms of the initial variable t, it suffices to take the integral

$$t = \int e^{\mu x_n} ds$$

Assertion 1 is well-known in the case when $\nu = 0$ (μ can then be any function of x). If $[u, v] = \mu v$, the phase flux of system (1.2) transforms the trajectories (1.1) into trajectories of the same system. The field u can therefore also be regarded as a symmetry field of system (1.1). When proving Theorems 1 and 2, we spoke only of the properties of the trajectories (and not of the solutions) of system (1.1). The theorems therefore also hold in the case of generalized symmetries.

Let us return to Eqs.(4.1) and consider whether there is a field u_ε (given by Eqs.(4.2)) which satisfies the relation $[u_\varepsilon, v_\varepsilon] = \mu v_\varepsilon + \nu u_\varepsilon$, where $\mu = \mu_0 + \mu_1\varepsilon + \ldots$, $\nu = \nu_0 + \nu_1\varepsilon + \ldots$ are series in powers of ε with constant coefficients. Eqs.(4.3) and (4.4) are replaced by the more general equations

$$\Sigma \frac{\partial Y_j^\circ}{\partial x_l}\omega_l = \nu_0 Y_j^\circ, \quad 1 \leqslant j \leqslant m \tag{7.6}$$

$$\Sigma \frac{\partial X_k^\circ}{\partial x_l}\omega_l = \Sigma \frac{\partial \omega_k}{\partial y_j} Y_j^\circ + \mu_0\omega_k + \nu_0 X_k^\circ, \quad 1 \leqslant k \leqslant n \tag{7.7}$$

We fix the coordinates $y_1, \ldots, y_m$ and consider in $\mathbf{T}^n = \{x_1, \ldots, x_n \bmod 2\pi\}$ the auxiliary system of equations $dx_k/ds = \omega_k = \text{const}$. We can then write (7.6) in the form $d(Y_j^\circ)/ds = \nu_0 Y_j^\circ$. Since the function Y_j° is periodic, it is bounded. Hence, either 1) $Y_j^\circ \equiv 0$, or 2) $\nu_0 = 0$.

In the first case, (7.7) takes the form

$$d(X_k^\circ)/ds = \nu_0 X_k^\circ + \mu_0\omega_k$$

Its solution (as a function of s) is $ce^{\nu_0 s} - \mu_0\omega_k/\nu_0$, $c = \text{const}$. If $\nu_0 \neq 0$, we have $c = 0$, since X_k° is bounded. In this case, $X_k \equiv -\mu_0\omega_k/\nu_0$, and the fields u_0 and v_0 are therefore linearly dependent.

Consider the second case, when $\nu_0 = 0$. If we assume "non-degeneracy" (if $\Sigma\omega_k\alpha_k \equiv 0$, $\alpha_k \in \mathbf{Z}$, then all the $\alpha_k = 0$), it again follows from (7.6) that the function Y_j° is independent of the variables x. On then averaging both sides of (7.7) with respect to $x_1, \ldots, x_n$, we arrive at the equations

$$\Sigma \frac{\partial \omega_k}{\partial y_j} Y_j^\circ + \mu_0\omega_k = 0, \quad k = 1, \ldots, n \tag{7.8}$$

We introduce the matrix $M = \| \partial\omega_k/\partial y_j \mid \omega \|$, where $\omega = \text{col}(\omega_1, \ldots, \omega_n)$. If $n \geqslant m$ and the rank of matrix M is almost everywhere $m + 1$, then we obtain from (7.8): $Y_j^\circ \equiv 0$ and $\mu_0 = 0$ (compare Para.4). Thus relations (7.7) take the form $\Sigma(\partial X_k/\partial x_l)\omega_l = 0$, whence it follows that the functions X_k° are independent of $x_1, \ldots, x_n$. Further, relations (4.5) are replaced by the following:

$$[i(\alpha, \omega) + \nu_1] g_\alpha = [i(\alpha, X^\circ) + \mu_1] f_\alpha, \quad \alpha \in \mathbf{Z}^n$$

The arguments of Para.4 apply in the case when ν is independent of ε. Then, $\nu_1 = 0$. If $y \in \mathbf{K}$, then $(\alpha, \omega) = 0$ and $f_\alpha \neq 0$. Consequently, $i(\alpha, X^\circ) + \mu_1 = 0$, whence we obtain simultaneously the equations $(\alpha, X^\circ) = 0$ and $\mu_1 = 0$. These relations enable us to extend Theorem 4 to the case of the commutation relation (7.1).

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Translated by D.E.B.

PMM U.S.S.R.,Vol.52,No.4,pp.420-426,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

STABILIZATION OF THE STEADY-STATE MOTIONS OF MECHANICAL SYSTEMS WITH CYCLICAL COORDINATES*

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The stabilization of the steady-state motions of holonomic systems with cyclical coordinates is considered, in cases when it is not essentially required that the system be exponentially stable with respect to all the phase variables. It is shown that the stabilization can be simplified by applying controls (of the feedback type) to only some of the cyclical variables. The control signals applied to the other cyclical variables are then used only to preserve the initial value of the momentum. From the initial equations, a linear subsystem which includes the controlled cyclical variables is isolated, and the methods of general control theory are used to construct control signals for it such that it is asymptotically stable with respect to the phase variables. Stability with respect to all the phase variables of the initial system is established by reducing the problem to a special case. When the subsystem has low dimensionality, the control coefficients can be found analytically, and when the dimensionality is high, they can be found by a computer with standard mathematical software, using the method of Repin and Tret'yakov /1/. The stabilization of systems with cyclical coordinates by applying forces with respect to these coordinates was first considered in /2/, from the standpoint of Lyapunov's second method /3/, and from the standpoint of general control theory /1/. The control signals were taken to be cyclical pulses, and asymptotic stability with respect to the positional coordinates and the velocities was obtained; it was remarked that control by forces applied with respect to the cyclical coordinates is possible. In /4, 5/, the stabilization of the steady-state motions of holonomic systems by forces applied with respect to the cyclical coordinates was analyzed qualitatively.

1. Consider a mechanical system which is constrained by geometrical non-stationary

*Prikl.Matem.Mekhan.,52,4,542-548,1988